JOURNAL OF ALGEBRA 137, 473–490 (1991)

Some Results in Local Rings on Ramification in Low Codimension

PHILLIP GRIFFITH*

Department of Mathematics, University of North Texas, Denton, Texas 76203, and Department of Mathematics, University of Illinois, Urbana, Illinois 61801

Communicated by Melvin Hochster

Received February 27, 1989

One way to study the behavior of the algebraic variety Spec A, where A is a normal domain, is to obtain finite maps f: Spec $A \to$ Spec R, where R is regular, and to investigate properties of Spec A relative to those of Spec R through the map f. In case A is finitely generated over a field k one can obtain such a map f via Noether normalization (see [19, p. 91]). In case A is local, complete, and contains a coefficient field k, one obtains an f by choosing (any) system of parameters $x_1, ..., x_n$ for A and then setting $R = k[[x_1, ..., x_n]]$ (see [19, pp. 210–212]; the case of unequal characteristic is slightly more delicate). Having constructed f: Spec $A \to$ Spec R by some means, the map f itself exerts a great deal of influence on the situation. In particular, the ramification which occurs in each fiber plays a central role in the interplay between properties of A and those of R. For example, should the map f be unramified one concludes that A must be regular and the map f must be étale. Moreover, the question of whether or not A/R is a ramified extension can be answered by computing ramification indices in codimension one. This is the essence of the theorem on "purity of branch loci" which can be found in [20, Sect. 41] or in [2].

When the regularity assumption on R is removed, then whether or not A/R is ramified cannot simply be detected in codimension one (e.g., see [10, Example 16.5, p. 85]). Nevertheless, one might expect to find "good" properties of R reflected in those of A in such a setting. For example, if R is a complete intersection (or perhaps just Gorenstein) and if A/R is unramified in codimension one, must A at least be Cohen–Macaulay?

In Section 1 of this article we obtain sufficient conditions in Theorem 1.4 (and resulting properties) for a complete local normal domain A to be a

* The author was partially supported by the National Science Foundation.

0021-8693/91 \$3.00

finite extension of a normal local domain B such that B is Gorenstein and A/B is unramified in codimension one (see Theorem 1.5 for the case where B is a complete intersection). In Section 2 we study the situation in which A/B is assumed to be an abelian normal extension as well as unramified in some specified low codimension. Two consequences of this investigation are: a positive result (Theorem 2.1) on purity is achieved in this setting, and the structure of these extensions in the complete case is described in a rather detailed and constructive fashion (see Theorem 2.2 and Theorem 2.4) under the assumption B contains appropriate roots of unity. In a sense Section 3 is devoted to a negative result with respect to the fundamental question posed in the previous paragraph. To be specific, we show how to start with any complete local normal domain A which contains an algebraically closed field of characteristic zero and to construct a module finite extension S such that S is a normal extension of a Gorenstein local normal domain B with S/B unramified in codimension one. Moreover, depth $S \leqslant$ depth A; thus S cannot be Cohen–Macaulay if A is not Cohen–Macaulay.

In Section 4 we consider a more special situation for A/B; namely, we assume the ring A is factorial. We expect better results in this context because of our results in [13]. In particular, it was established in [13, Theorem 2.1] that, if A is a local normal domain which is a normal extension of a regular local ring B such that the ramified primes in codimension one are principal, then A is a complete intersection. Moreover, in [13, Theorem 2.4] it is shown that such a complete local domain A is a normal extension of a special complete intersection D (in the sense of Theorem 1.5) such that A/D is unramified in codimension one. With these results as motivation, we consider in Section 4 a "test problem" for our theory which is a variant on a question posed by Samuel [22]:

Let A be a complete local factorial domain which is an isolated singularity (i.e., A_p is regular for nonmaximal prime ideals P) and which contains an algebraically closed coefficient field of characteristic zero. If A is unramified in codimension one over a complete intersection, must A be Cohen–Macaulay (hence Gorenstein)?

With regard to sources for our notation and terminology we suggest Bourbaki [6, Chap. VI (Sect. 8)] and Nagata [20, Chap. VI] for ramification theory, Fossum [10] for information on divisor class groups, and Matsumara [19] for standard facts in commutative algebra.

1. Normal Closures and Ramification Theory for Complete Local Domains

In part the material in this section is inspired by M. Auslander's [2] proof of the "purity of branch locus" theorem in which he reduced the problem to the case where the extension of fraction fields is a Galois extension. We will describe our version of this idea below.

Let A be a local normal domain which is a module finite extension of an excellent, local normal domain R and such that the extension of residue fields is trivial. In case A is complete, regular local R's are guaranteed from the Cohen structure theory (see [18, p. 212]). Let K and L denote the fraction fields for R and A, respectively. In order for our techniques to succeed we need that $[A:R]=[L:K]$ represents a unit in R. The *normal closure* (not to be confused with integral closure of an integral domain) of the ring extension A/R is defined as follows. Let E/K be the Galois closure of the field extension L/K (note that L/K will be separable since $[L:K]$ is a unit in K). In other words the field E is the smallest Galois extension of K which contains L. Further, let S be the integral closure of A in E. Since A is a module finite extension of an excellent local ring R, then S is known to be a module finite extension of A. Moreover, if A is complete then S is necessarily a complete local normal domain (see [19, Sect. 32]). We will refer to the ring extension S/R as being the *normal closure* of the extension A/R. We remark that, if A contains a normal domain D such that D is a normal extension of R (i.e., D/R is its own normal closure), then the normal closure of A/R contains the normal closure of A/D. Normal closures naturally give rise to Galois groups. Specifically there is a group $G=\text{gal}(E/K)$ and a subgroup $H=\text{gal}(E/L)$ such that $S^G=R$ and $S^H=A$. Our main situation of interest is that in which R is regular local (or complete regular local). In this case the minimum degree $[A:R]$ (equivalently, the module rank of A over R) which can be attained is the multiplicity e_A of A (see Serre [23, Chap. 5]). Finally, we mention that all induced extensions of residue fields of maximal ideals are trivial if this is the case for the initial extension A/R.

Before starting with our first result on normal closures, we wish to make a few remarks on ramification in ring extensions. Let R be a ring and let A be an R-algebra. A prime ideal P of A is said to be unramified over R provided the R-ideal $p=P\cap R$ satisfies:

(a) $pA_P=PA_P$ and

(b) A_P/pA_P is a separable field extension of R_P/pR_P.

One says that A is unramified over R provided each prime ideal of A is unramified over R (it is sufficient that each maximal ideal in A be unramified over R (see [19, p. 134])). We use the expression "A is unramified over R in codimension i" to mean that each prime P of A with height $P\leqslant i$ is unramified over R. Within this context we note the multiplicative property of ramification; namely, A/B and B/R are unramified in codimension i if and only if A/R is unramified in codimension i (see

Auslander and Buchsbaum [3, Proposition A.2]). In case A/R is a finite extension of normal domains which is unramified in codimension one, then any codimension one prime P in R has a primary decomposition $pA = \bigcap_{i=1}^{t} P_i$, where the primes P_i represent the distinct primes in A which contract to p. Moreover, if the extension A/R is normal with Galois group G, then the primes P_i are G-conjugate to one another.

Our initial observation which connects the two topics discussed above was pointed out in [13, Corollary 1.2] and is due to Auslander and Goldman [4, Proposition A.4].

(1.1). A normal extension A/R of the normal domains A and R is unramified in codimension one if and only if $\mathrm{Hom}_R(A, R)$ is generated as an A-module by the trace map of A over R.

The following rather technical result represents a slight generalization of an observation made by Auslander [2, p. 117] which in part he attributes to Abyankar [1]. To be precise, the import of (1.2) is a restatement of Auslander's observation in the situation $R = B$.

PROPOSITION 1.2. *Let R be an excellent normal domain and let B/R be a normal extension of R. Further, let A be a normal domain which is a module finite extension of B such that $[A:B]$ is a unit in B. Let S/R be the normal closure of A/R. If A/B is unramified in codimension one, then the same is true of the extensions S/B and S/A.*

Proof. Since establishing the result for S/B will ensure the result for S/A we will focus our attention on S/B.

If $S = A$ of course we are finished; so we proceed by induction on $[S:A]$. With this in mind together with the observation that S is the integral closure of the R-algebra generated by the composite of the G-conjugates of A (here $G = \mathrm{gal}(S/R)$) and that each of these G-conjugates contains B since B/R is a normal extension, one sees that it suffices to show that if C and D are two B-algebras in S such that each is a normal domain which is unramified over B in codimension one, then the integral closure of the composite CD is also unramified over B in codimension one.

In order to achieve the above goal we first observe that we may replace B by $I = C \cap D$ since both C and D are unramified in codimension one over I as well. Furthermore, since the extension D/I is unramified in codimension one, we have that the (base change) extension $C = C \otimes_I I \to C \otimes_I D$ is also unramified in codimension one (recall that in codimension one both C and D are free over I). Moreover, since C is normal it follows that $C \otimes_I D$ is regular in codimension one; that is, $C \otimes_I D$ satisfies the condition R_1. It follows that the ring $C \otimes_I D$ has the property that its reflexive closure, $(C \otimes_I D)^{**}$, with respect to R is a normal ring. Hence $(C \otimes_I D)^{**}$ decomposes as a direct product of complete local, normal domains, each of which is unramified in codimension one over I. These summands represent the various composites of C and D in the algebraic closure of the fraction field of I. In particular, one of them represents the integral closure $(CD)'$ of the composite CD. The induction bridge is now complete.

An immediate consequence of the above statement is the following statement concerning normal closures of isolated singularities.

COROLLARY 1.3 (same setup as in 1.2). *If A satisfies the condition R_i (i.e., regular in codimension $\leqslant i$), then so does S. In particular, if A is an isolated singularity, then so is S.*

Proof. The conclusion here follows from the fact that S/A is unramified in codimension one (a local property) and the theorem on the purity of branch locus (see introduction of [19, p. 158]).

Proposition 1.2 and Corollary 1.3 have application in the following context. We suppose that A is a complete local normal domain and as such A is a module finite extension of a complete regular local ring R. We suppose that $[A:R]$ is a unit in R and, as usual, the extension of residue fields is trivial. We further suppose that A contains an R-algebra B such that B/R is a normal extension and such that A/B is unramified in codimension one. Then from Proposition 1.2 we conclude that the normal closure S/R of A/R has the property that S/B and S/A are unramified in codimension one. Then Corollary 1.3 gives that S is an isolated singularity if A is. Our next two results suggest when such a B exists with the property that B is a Gorenstein ring. The notation $\mathrm{tr}_{A/B}$ indicates the reduced trace of A over B, i.e., $\mathrm{tr}_{A/B}(1)=1$.

Before getting into the statements of (1.4) and (1.5) we remind the reader of the notion of dualizing module. Let A be a local ring which is a module finite extension of a Gorenstein local ring R. The A-module $\Omega_A = \mathrm{Hom}_R(A, R)$ is called the "dualizing module" (or "canonical module") for A. (See Grothendieck [16] or Herzog and Kunz [18] or [9, pp. 9–10] for special properties and equivalent constructions.) In case A is Cohen–Macaulay then Ω_A has finite injective dimension and in case A is Gorenstein then $A \cong \Omega_A$. Moreover, Ω_A is always a divisorial ideal in the situation where A is a normal domain; that is, Ω_A represents an element in the divisor class group of A under this assumption.

THEOREM 1.4. *Let A be a complete local normal domain that is a normal extension of the regular local ring R. We further suppose that R contains the nth roots of unity, where $n=[A:R]$. If the dualizing module of A is cyclic then there is a Gorenstein local domain B in A/R such that B is an abelian* *normal extension of R and such that A/B is unramified in codimension one. Conversely, if A/B is unramified in codimension one with B Gorenstein, then the dualizing module for A is cyclic.*

Proof. The extension of residue fields is necessarily separable since $[A:R]$ is a unit (the degree of the field extension is a divisor of $[A:R]$). Thus, we may assume that R has been extended in A to accommodate the residue field extension; that is, without loss of generality we may assume the residue field extension from R to A as trivial.

Next we suppose that a Gorenstein local domain B exists with A/B unramified in codimension one; hence $\mathrm{Hom}_B(A, B)$ is cyclically generated as an A-module by the trace map $\mathrm{tr}_{A/B}$. However, since B is Gorenstein and A is a finitely generated B-module, it follows that $\mathrm{Hom}_B(A, B)$ is isomorphic to the dualizing module for A.

Finally, we suppose that the normal extension A/R has the property that the dualizing module Ω_A is cyclic. Since A is finitely generated over the regular local ring R we conclude as above that $\Omega_A \cong \mathrm{Hom}_R(A, R)$; hence there is an R-map $\pi: A \to R$ such that $\mathrm{Hom}_R(A, R) = A\pi$. The remainder of our argument now proceeds along the lines of our argument given in [13, pp. 472, 473]. First we note that the trace map $\mathrm{tr}_{A/R}$ is in $\mathrm{Hom}_R(A, R)$ and hence that $\mathrm{tr}_{A/R} = b\pi$ for some $b \in A$. Since the module $\mathrm{Hom}_R(A, R)$ is a G-equivariant module for $G = \mathrm{gal}(A/R)$ and since G treats $\mathrm{tr}_{A/R}$ as an invariant in $\mathrm{Hom}_R(A, R)$ (see [13, p. 467] for the G-structure of $\mathrm{Hom}_R(A, R)$), we have as in [13, pp. 472, 473] that the action of G on b induces a one-cocyle in $H^1(G, V_A)$, where V_A denotes the group of units in A. However, [13, Lemma 2.3] allows us to multiply b by a suitable unit in A (and adjusting π likewise by the inverse of the unit) so that $b^l \in R$ for some $l \geqslant 1$. It follows that the extension B/R, where B is the integral closure of $R[b]$, is a cyclic normal extension. From P. Roberts' main theorem [21] we have that B is a Cohen–Macaulay local domain. Since the extension B/R is normal, there is a normal subgroup H of G such that $A^H = B$. Moreover, since π is an element in $\mathrm{Hom}_R(A, R)^H \cong \mathrm{Hom}_R(B, R)$ it follows that $\mathrm{Hom}_R(B, R)$ is generated by the restriction π' of π to B. Finally, as the argument [13, p. 473] shows, one gets that $\pi = \pi' \circ \mathrm{tr}_{A/B}$ from which one concludes via the adjoint isomorphism theorem

$$\mathrm{Hom}_B(A, B) \cong \mathrm{Hom}_B(A, \mathrm{Hom}_R(B, R)) \cong \mathrm{Hom}_R(A, R)$$

that $\mathrm{tr}_{A/B}$ generates $\mathrm{Hom}_B(A, B)$ as an A-module.

In order to illustrate the nature and location of the ring B we consider the following example.

(1.4a). Let k be an algebraically closed field of characteristic other than 2. Let $A = k[[X, Y]]$ and $R = k[[X^4, Y^4]]$. Since A is regular local its dualizing module $\Omega_A \cong \mathrm{Hom}_R(A, R)$ is certainly cyclic. As an R-module A is necessarily free and, in particular, A has a basis given by the sixteen elements $\{X^i Y^i \mid 0 \leqslant i, j \leqslant 3\}$. The R-homomorphism $\pi: A \to R$ defined by

$$\pi(X^i Y^i) = \begin{cases} 0, & \text{if} \quad i \neq 3 \quad \text{or} \quad j \neq 3 \\ 1, & \text{if} \quad i = j = 3 \end{cases}$$

is easily seen to generate $\mathrm{Hom}_R(A, R)$ as an A-module. Moreover one observes that $\mathrm{tr} = (X^3 Y^3)\pi$, where $\mathrm{tr}: A \to R$ is the reduced trace map. Hence we may take $b = X^3 Y^3$ as in the proof of Theorem 1.4. It follows that the local ring B is equal to the integral closure of $R[X^3 Y^3]$, which in this instance is isomorphic to the hypersurface $k[[U, V]][T]/(T^4 - UV)$.

The next result describes how one may find Gorenstein rings B as above in case the extension A/R is not normal; hence an automorphism group $G = \mathrm{gal}(A/R)$ is not present. In order to handle this case we need to define a notion which we refer to as *uniform ramification.* To be precise, we say that the prime P in the R-algebra A is uniformly ramified provided the ramification index of each prime Q in A having the same codimension as P and in the fiber of $p = P \cap R$ has also the same ramification index as P. If each fiber of A/R has this property for some particalar codimension i, then we say that A/R is uniformly ramified in codimension i. That is, A/R is uniformly ramified in codimension i provided the ramification index is constant on each fiber in codimension i. As in the case of Theorem 1.4, our proof of Theorem 1.5 will follow closely our argument in [13, pp. 473–474 (Theorem 2.4)]. For the sake of completeness, we include most of the argument here.

THEOREM 1.5. *Let the local normal domain A be a module finite extension of the regular local ring R such that the extension of residue fields is trivial, $n = [A : R]$ is a unit, and such that R contains the nth roots of unity. If the ramified primes in A/R of codimension one are principal and if the ramification in codimension one is uniform, then there is a complete intersection D in A/R such that D/R is an abelian normal extension and such that A/D is unramified in codimension one.*

Proof. Let $p_1, \ldots, p_t$ be the distinct principal primes in R which have positive ramification in their fibers. From our hypothesis we have that $p_i = (q_{i1} \cdots q_{i\Delta(i)})^{l_i}$, for $i = 1, \ldots, t$, where the q_{ij} represent the distinct principal primes in A which ramify over R. Let $d_i = q_{i1} \cdots q_{i\Delta(i)}$ for each i. Then the minimum polynomial for d_i over R is $X^{l_i} - p_i$. Moreover, since l_i divides $n = [A : R]$ and since R contains the nth roots of unity, the extension $K[d_i]/K$ is cyclic Galois, where K is the fraction field of R. Since l_i is necessarily a unit in R it follows that $D_i = R[d_i]$ is integrally closed in

$K[d_i]$. Furthermore, the only codimension one prime in D_i which ramifies over R is (d_i) (note that (d_i) is prime in D_i since $D_i/(d_i) \cong R/(p_i)$). It follows that $D_i \cap D_j$ is unramified in codimension one for each $i \neq j$. Since the residue field extension is trivial we have from the purity of branch locus theorem that $D_i \cap D_j = R$ for $i \neq j$; that is, D_i and D_j are linearly disjoint in R for $i \neq j$. Both D_i and D_j are free R-modules; hence the ring $D_i \otimes_R D_j$ is normal and isomorphic to the composite $D_i D_j$, for $i \neq j$. Arguing by induction we get that $D_1 \cdots D_i$ and D_{i+1} are linearly disjoint and normal abelian extensions of R. Thus, so is $D_1 \cdots D_{i+1}$. In particular, the composite $D = D_1 \cdots D_t \cong D_1 \otimes_R \cdots \otimes_R D_t$ is an abelian normal extension of R which is isomorphic to $R[X_1, ..., X_t]/(X_1^{l_1} - p_1, ..., X_t^{l_t} - p_t)$. Finally, the multiplicative property of ramification gives that the principal primes $q_{ij}A$ are unramified over d_iD. Thus, the extension A/D is unramified in codimension one.

As an application of the preceding results we are able to give a reasonably nice answer to the question of when a finite extension A/R of regular local rings is a normal extension (i.e., when the extension of fraction fields is Galois). Of course one needs that the extension of fraction fields is separable, which is ensured if $[A:R]$ is assumed to be a unit in R. Also, if we are to apply Theorem 1.5, we must assume that the nth roots of unity are in R where $n = [A:B]$. Moreover, if there is a finite group G such that $A^G = R$, then the ramification in codimension one must be uniform since all primes in A lying over a fixed prime in R are G-conjugate (see [18, pp. 33, 34]). The forthcoming statement says that this condition is also sufficient in case the rings are complete.

THEOREM 1.6. *Let A/R be a module finite extension of complete regular local rings such that the extension of residue fields is trivial, $n = [A:R]$ is a unit in R, and such that R contains the nth roots of unity. Then A/R is a normal extension if and only if A/R is uniformly ramified in codimension one.*

Proof. If A/R is a normal extension then as observed above it follows from [18, pp. 33, 34] that A/R is uniformly ramified in codimension one. So now assume that A/R satisfies the uniform ramification condition. Since A is necessarily factorial it follows from Theorem 1.5 that there is a complete intersection D in A/R such that D is an abelian normal extension of R and such that A/D is unramified in codimension one. Let S/R be the normal closure of A/R. It follows from Proposition 1.2 that S/A is unramified in codimension one. We now employ the purity of branch locus theorem to obtain that S/A is unramified. Of course S is local since it is a module finite extension of the complete local normal domain A. Moreover the residue field extension from A to S is trivial. It then follows that $A = S$ and hence that A/R is a normal extension.

In order to illustrate Theorem 1.6 we consider the following elementary example.

(1.6a). Let k be an algebraically closed field of characteristic $\neq 2$ and let $A = k[[X, Y]]$, $R = k[[X^2, Y^2]]$, and let $B = k[[X^2, XY, Y^2]]$. Then A/B is unramified in codimension one (see [10, p. 85]) where $B \cong k[[U, V, W]]/(U^2 - VW)$ (see Example 2.3) is a hypersurface. Moreover, B/R is a cyclic normal extension of degree two and A/R is a normal extension of degree 4.

2. Purity Criteria for Certain Abelian Normal Extensions

Some natural questions which arise in the study of ring extensions that are unramified in some predetermined (low) codimension are:

(1) Is there a "significant" generalization of the theorem on purity of branch locus for (essentially) finite extensions A/R in which R is regular local to the case when R is assumed to be a complete intersection, or merely Gorenstein?

(2) Even if a good result on purity cannot be obtained in the context of (1), is it possible that A/B unramified in codimension one, for B a complete intersection (or simply Gorenstein), ensures that A is at least Cohen–Macaulay?

With regard to question (1) above, one may combine Grothendieck's result [15] on parafactoriality for complete intersections together with Roberts' [21] circle of ideas on abelian extensions in order to obtain a reasonably clean result on purity. This is recorded in Theorem 2.1. As for question (2) we demonstrate via Theorem 3.1 that, at least as stated, question (2) has a negative answer for B Gorenstein.

Theorem 2.1. *Let B be an excellent, local normal domain which is a complete intersection and suppose the local normal domain A is an abelian normal extension of B such that $[A : B]$ is prime to the characteristic of the residue field of B and such that the induced extension of residue fields is trivial. If either*

(i) *B is factorial and A/B is unramified in codimension one, or if*

(ii) *A/B is unramified in codimension three, then A/B is unramified. In particular A is Cohen–Macaulay.*

Proof. The assumption of excellence allows us to pass to the situation in which A and B are complete local rings. Then following Robert's [21] argument we see that $A = \bigoplus_{j=1}^{n} I_j$ where the decomposition is over B and where the I_j are divisorial B-ideals. This is a result of lifting the group algebra structure of A/mA to A where m is the maximal ideal of B. This brings us to the heart of most purity arguments; namely we can reach our desired conclusion provided we can establish that A is free as a B-module. However, in the case of hypothesis (i) one immediately gets that A is free as a B-module. Under hypothesis (ii) one gets that the divisorial B-ideals I_j are necessarily principal in codimension three. However, this observation puts us into the thrust of Grothendieck's theorem [15] on parafactoriality of a complete intersection and allows us to conclude that A is a free B-module. Having established that A is free over B we now proceed as in Auslander and Buchsbaum [3, Corollary 2.7] and Nagata [20, Sect. 41.5] in order to establish that A/B is unramified.

In the course of studying abelian normal extensions the above line of thought is suggestive for a recipe in constructing such extensions. This we accomplish in Theorem 2.4. Our next result points out the restrictions on the divisorial ideals which are necessary for the construction in 2.4.

THEOREM 2.2. *Let A/B be an abelian extension of complete local normal domains. If A/B is unramified in codimension one, then $A \cong \bigoplus_{j=1}^{n} I_j$ where the I_j are divisorial ideals of B such that the set of isomorphism classes of these ideals forms a subgroup of $\mathscr{C}l B$.*

Proof. The argument of Roberts [21] as given in Theorem 2.1 applies once again; namely, $A \cong \bigoplus_{j=1}^{n} I_j$ where the I_j are divisorial ideals in B. Of course B itself is one of them, as given by the image of the trace map.

As pointed out in M. Auslander's article [2, p. 118] there is a natural isomorphism of A-modules $\Delta(A; G) \cong \operatorname{Hom}_B(A, A)$, where $\Delta(A; G)$ is the "twisted" or skew group algebra. It follows that $\operatorname{Hom}_B(A, A)$ is free as an A-module. As a consequence we have for each j and k that $\operatorname{Hom}_B(I_j, I_k)$ is again one of the divisorial B-summands of A. We note that $I_j^* \cong \operatorname{Hom}_B(I_j, B)$ is one of these as well, since B itself is a summand of A via the trace map. Thus, using the fact that the product in the divisor class group of B may be defined via

$$[I] \cdot [J] = [\operatorname{Hom}_B(I^*, J)]$$

(see [10] or [13, p. 81]), it follows that $[I_j][I_k] = [\operatorname{Hom}_B(I_j^*, I_k)]$ and hence that each element in the subgroup of $\mathscr{C}l B$ generated by the $[I_j]$ has a divisorial representative as a B-summand of A.

Before describing how one actually constructs such extensions we recall a rather simple example in dimension two (see Example (1.6a)) which shows that Theorem 2.1 cannot be weakened too much.

EXAMPLE 2.3 (or (1.6a) revisited). Let k be a field of characteristic other than 2 and let $A = k[[X, Y]]$. Then A is a quadratic extension of the complete local normal domain $B = k[[X^2, Y^2, XY]] \cong k[[U, V, W]]/(U^2 - VW)$. The ring A is unramified over B in codimension one (see [10, p. 85]). However, A is not unramified over B since A is regular and B is not. It is known that the divisor class group of B is isomorphic to $\mathbb{Z}_2$ with generator the class of $I = (X^2, XY)$; note $I^2 = X^2(X^2, XY, Y^2)$ and hence that $I^{(2)} = (X^2)$. Thus, in view of Theorem 2.2, it follows that $A \cong B \oplus I$ as a B-module.

THEOREM 2.4. *Let B be a complete local normal domain having a divisorial ideal I of finite order n in the divisor class group of B. We assume that n is prime to the characteristic of the residue field of B and that B contains the nth roots of unity. Let $a \in B$ such that $I^{(n)} = (a)$ and let A be the integral closure of B in the field extension $K[\sqrt[n]{a}]$, where K is the fraction field of B. Then A/B is a cyclic extension of order n which is unramified in codimension one. Moreover, A has the form*

$$A = \bigoplus_{j=0}^{n-1} I^{(j)} \frac{\sqrt[n]{u^j}}{b^j},$$

where $au = b^n$ in K.

Proof. If the element a should have an lth root in B where l divides n, then one would conclude that $I^{(e)} = (\sqrt[l]{a})$ where $e = n/l$. This would contradict that the order of I is n should $l > 1$. It follows that $X^n - a$ is irreducible in both $B[X]$ and $K[X]$ and that $K[\sqrt[n]{a}]$ is a cyclic Galois extension of order n (recall that B contains the nth roots of unity). Moreover, A is a complete local normal domain since it is necessarily a module finite extension of B.

Let $b \in B$ such that $I_p = (b)_p$ for the finite number of primes P in the primary decomposition of I. Let S denote the complement in B of this finite number of prime ideals of codimension one. We have that $au = b^n$ where $u \in K$ represents a unit in $S^{-1}B$. The relation $au = b^n$ gives that $K[\sqrt[n]{a}] = K[\sqrt[n]{u}]$. Moreover, the requirement that an element $z = \sum_{j=0}^{n-1} c_j(\sqrt[n]{u^j}/b^j)$ be integral over B imposes constraints on the elementary symmetric polynomials evaluated at the conjugates of z (i.e., the coefficients of the polynomial $\prod_{j=0}^{n-1} (X - \sigma^j(z))$ must be in B where $\sigma(\sqrt[n]{u}/b^j) = \zeta(\sqrt[n]{u}/b^j)$ with ζ a primitive nth root of unity) which yield that $c_j \in I^{(j)}$ for each j. It follows that $A = \bigoplus_{j=0}^{n-1} I^{(j)}(\sqrt[n]{u^j}/b^j)$. Finally, it remains to argue that A/B is unramified in codimension one.

We begin this task by noticing that the ring $S^{-1}A$ represents the integral closure of $X^n - u$ in $K[\sqrt[n]{a}] = K[\sqrt[n]{u}]$. From [8, p. 114 (Exercise 8)], we have that $X^n - u$ is a separable polynomial in $S^{-1}B[X]$ and hence that $S^{-1}B \to S^{-1}A$ is an unramified extension. This takes care of all primes in B of codimension one which contain a. If Q is a prime ideal in B such that $a \notin Q$ then $X^n - a$ is a separable polynomial in $B_Q[X]$ for the same reason as above (namely, $X^n - a$ has distinct roots modulo the maximal ideal of B_Q). Again $B_Q \to B_Q \otimes_B A$ is unramified. It now follows that A/B is unramified in codimension one.

COROLLARY 2.5 (notation as in 2.4). *The divisorial ideal I has its divisor class $[I]$ in the kernel of the group homomorphism $\mathscr{C}lB \to \mathscr{C}lA$, i.e., $(IA)^{**} = \sqrt[n]{a}\,A$ where $(\_)^* = \operatorname{Hom}_A(\_, A)$.*

Proof. There is an A-homomorphism $\sqrt[n]{a}\,A \to (IA)^{**}$ which is an isomorphism in codimension one. Since A is normal this map must be an isomorphism.

We recall from Section 1 (see discussion preceding Theorem 1.4) that the dualizing module Ω_A for a local normal domain A represents an element in the divisor class group of A.

THEOREM 2.6. *Let A be a complete local normal domain which contains an algebraically closed coefficient field and suppose its dualizing module Ω_A has finite order n as an element in $\mathscr{C}lA$. We assume that n is prime to the characteristic of A. Then there is a cyclic normal extension S/A such that $S \cong \Omega_S$ and such that S/A is unramified in codimension one.*

Proof. We apply Theorem 2.4 and Corollary 2.5 in order to construct a normal cyclic extension S/A such that S/A is unramified in codimension one and such that the class of Ω_A is in the kernel of the group homomorphism $\mathscr{C}lA \to \mathscr{C}lS$. It remains to show that $S \cong \Omega_S$. This follows immediately from the following lemma.

LEMMA 2.7. *Let S/A and A/R be module finite extensions of local normal domains in which R is assumed to be Gorenstein and S/A is assumed to be unramified in codimension one. Then*

$$\Omega_S \cong (S \otimes_A \Omega_A)^{**},$$

where $(\_)^* = \operatorname{Hom}_S(\_, S)$.

Proof. We note that such an R always exists for the extension S/A in case they are complete.

One obtains a map $S \times \operatorname{Hom}_R(A, R) \to \operatorname{Hom}_R(S, R)$ via $(\delta, f) \mapsto f_\delta$, where $f_\delta(x) =: f(\operatorname{tr}(\delta x))$ and where $\operatorname{tr}: S \to A$ is the reduced trace map. This induces an S-homomorphism $S \otimes_A \operatorname{Hom}_R(A, R) \to \operatorname{Hom}_R(S, R)$, which is necessarily an isomorphism in codimension one (recall from (1.1) that $\mathrm{Hom}_A(S, A) = S \cdot \mathrm{tr}$) since S/A is unramified in codimension one. Identifying Ω_A with $\mathrm{Hom}_R(A, R)$ and Ω_S with $\mathrm{Hom}_R(S, R)$ and applying the fact that S is normal then completes the argument.

COROLLARY 2.8. *Let A/R be a normal extension of complete local normal domains in which R is factorial and Gorenstein and has an algebraically closed coefficient field of characteristic zero. Then there is a cyclic normal extension S/A which is unramified in codimension one such that $S \cong \Omega_S$.*

Proof. From [14, Theorem 2.1] we have that $\Omega_A \cong \mathrm{Hom}_R(A, R)$ is a G-module, where $G = \mathrm{gal}(A/R)$, and therefore has finite order in $\mathscr{C}lA$. Hence the existence of S follows from Theorem 2.6.

We conclude this section by noting that Theorem 2.4 provides a format for constructing normal extensions A of complete intersections B such that A/B is unramified in codimension one and such that A is not Cohen–Macaulay. According to Theorem 2.4 one simply needs a normal complete intersection B having divisorial ideal I such that I is not Cohen–Macaulay as a B-module and such that the class of I is a torsion element in $\mathscr{C}lB$. While there are many examples of non-Cohen–Macaulay divisorial ideals for normal complete intersections (e.g., see the article [14, Theorem 1.3] for the case of quadratic hypersurfaces) we are not at the time of this writing aware of any that represent torsion elements in the divisor class group.

3. ABUNDANCE OF NORMAL EXTENSIONS OF GORENSTEIN RINGS WHICH ARE UNRAMIFIED IN CODIMENSION ONE

Throughout this section we shall only consider local rings which contain an algebraically closed field of characteristic zero. Let A be such a local ring which is a complete normal domain. Then A contains a complete regular local ring R such that A/R is a module finite extension. Let C/R be the normal closure of A/R. By [14, Theorem 2.1] the dualizing module Ω_C of C represents an element of finite order in the divisor class group of C. From Corollary 2.8 there is a cyclic normal extension T/C such that T/C is unramified in codimension one and such that $\Omega_T \cong T$. Let S/R be the normal closure of T/R. With the aid of Proposition 1.2 we see that S/C and S/T are unramified in codimension one. Since $\Omega_T \cong T$ it follows from Lemma 2.7 that $\Omega_S \cong S$. By Theorem 1.4 there is an intermediate cyclic extension B/R in S/R such that B is Gorenstein and such that S/B is unramified in codimension one. By considering the various trace maps one sees that the depth of A must be at least as large as depth S. In particular the ring S is not Cohen–Macaulay if A is not Cohen–Macaulay. We summarize these observations in the following statement.

THEOREM 3.1. *Let A be a complete local normal domain which contains an algebraically closed field of characteristic zero. Then there is a normal extension S/A such that S is also a normal extension of a Gorenstein local ring B with S/B unramified in codimension one. The Gorenstein ring B is the integral closure of a cyclic normal extension of a regular local ring. Furthermore, the depth of S cannot exceed the depth of A.*

COROLLARY 3.2 (notation as above). *Let A be as in* 3.1 *but with the additional assumption that A itself is a normal extension of a regular local ring. Then S/A can be constructed with the additional property that S/A is unramified in codimension one.*

Proof. In the argument preceding 3.1 we may take $A = C$ since the dualizing module for A will represent a torsion element in $\mathscr{C}lA$ in this case.

The above results demonstrate that all complete normal local domains (at least in equal characteristic zero) are closely related to ones which are unramified extensions in codimension one over some Gorenstein ring B. Among other things these results give a rather convincing negative answer to the second question posed at the outset of Section 2. Nevertheless it is not clear when (or if) B could have the more restrictive property of being a complete intersection.

4. Some Structural Results for Complete Local Factorial Domains of Equal Characteristic Zero

In this section we take up the "test problem" described in our introduction. Our starting point is to establish a generalization of [13, Theorem 2.1] of sorts in the context of *supernormal* domains, that is, integral domains which satisfy the conditions S_3 and R_2 (e.g., isolated singularity of depth $\geqslant 3$). From the discussion in Hartshorne and Ogus [17] one sees that supernormal domains are less rare in equal characteristic zero (see also the discussion of rings of equal characteristic zero that have discrete divisor class groups (DCG) in Danilov [7] or in Fossum [10, pp. 124–130]).

THEOREM 4.1. *Let A be a complete local supernormal domain which contains an algebraically closed field of characteristic zero. Let A be a module finite extension of a complete regular local ring R for which the extension of residue fields is trivial. If the ramification in A/R satisfies*:

(1) *A/R is uniformly ramified in codimension one*

(2) *the ramified primes of codimension one in A are both principal and normal,*

(3) *the normal closure S/R of A/R has the property that S satisfies the condition S_3,*

then S is a complete intersection and A is Gorenstein.

Proof. From Theorem 1.5 there is a complete intersection D such that $R \subseteq D \subseteq A$, such that D/R is a normal extension and such that A/D is unramified in codimension one. By Proposition 1.2 it follows that the normal closure S/R of A/R has the property that S/D and S/A are unramified in codimension one. From Corollary 1.3 and the theorem on purity of branch locus, we get that S satisfies R_2 and that S/A is unramified in codimension two. In particular S is a supernormal complete local domain. Let p be a principal prime in A which ramifies over R. From the above discussion and properties of base change, we see that $A/pA \to S/pS$ is unramified in codimension one and as such the factor ring S/pS satisfies R_1. Moreover, since S is supernormal we also have that S/pS satisfies the condition S_2: thus S/pS is a normal ring. Hence, if the prime ideal (p) of A should split nontrivially in S, it would follow that S/pS is a nontrivial direct product of local domains. However, S/pS is necessarily local since S is. Thus, the only way out of this conflict is that the prime (p) cannot split in S. Therefore, since (p) cannot split or ramify in S it follows that pS is a principal prime ideal in S. From [13, Theorem 2.1] we obtain that S is a complete intersection. Since A is necessarily an R-summand of S we conclude that A is Cohen–Macaulay. Further, since A/D is unramified in codimension one it follows that $A \cong \mathrm{Hom}_D(A, D)$ is the dualizing module for A; thus, A is a Gorenstein ring.

Our main structural result on complete factorial domains now follows.

THEOREM 4.2. *Let A be a complete local factorial domain having an algebraically closed coefficient field of characteristic zero and suppose that A is an isolated singularity. Further, let A be a module finite extension of a complete regular local ring R. If the extension A/R is uniformly ramified in codimension one, then A/R contains a unique maximal normal extension D/R such that D is a factorial complete intersection and such that A/D is unramified in codimension one. Moreover, A is a normal extension of R if and only if $A = D$.*

Proof. As usual we may reduce to the case that the residue field extension from R to A is trivial, since all of the hypotheses remain unchanged for R enlarged in this fashion. By Theorem 1.5 there is an abelian normal extension D_0/R in A/R such that D_0 is a complete intersection and such that A/D_0 is unramified in codimension one. Let S/R be the normal closure of A/R. Let G be the Galois group of S/R and let H be the same for the extension S/A, i.e., $S^H = A$. Since S/A is unramified in codimension one, by Proposition 1.2, and since A is factorial it follows from [13, Lemma 2.3] that

$$H^1(H, U_S) = \operatorname{Hom}(H, k^*) = \{1\},$$

since

$$H^1(H, U_S) \cong \operatorname{Ker}(\mathscr{C}lA \to \mathscr{C}lS) = 0$$

(see [10, Theorem 16.1]), where U_S and k^* denote the multiplicative groups of units of S and its residue field k, respectively. Thus, the subgroup H is equal to its own commutator group H' (recall that k^* contains all roots of unity). Let J be the subgroup of G which is generated by all conjugates of H in G. The subgroup J has the following two properties:

(1) J is equal to its own commutator subgroup J'; hence $J < G'$, and

(2) J is a normal subgroup of G.

Statement (1) holds since a nontrivial homomorphism from J to an abelian group V would necessarily induce a nontrivial homomorphism from some conjugate of H (and hence H itself) to V. As for statement two one sees that the equality

$$\begin{aligned} &x^{-1}(g_1^{-1}h_1 g_1 \cdot g_2^{-1}h_2 g_2 \cdots g_n^{-1}h_n g_n)x \\ &\quad = (g_1 x)^{-1} h_1(g_1 x)(g_2 x)^{-1} h_2(g_2 x) \cdots (g_n x)^{-1} h_n(g_n x), \end{aligned}$$

for $x, g_1, ..., g_n \in G$ and $h_1, ..., h_n \in H$, indicates that J is normal in G.

We let $D = S^J$. Then $R \subseteq D \subseteq A$. Since S/A is unramified in codimension one we have that $S \cong \operatorname{Hom}_A(S, A)$. However, the fact that A is factorial yields that $A \cong \operatorname{Hom}_R(A, R)$ and thus, from the adjoint isomorphism theorem, gives that $S \cong \operatorname{Hom}_A(S, A) \cong \operatorname{Hom}_A(S, \operatorname{Hom}_R(A, R)) \cong \operatorname{Hom}_R(S, R)$. We employ Theorem 1.4 to obtain that there is an abelian normal extension B/R in S/R with B Gorenstein and with S/B unramified in codimension one. Now returning to D we see that necessarily $B \subseteq D$ since the Galois group of B/R is abelian and since $J = J' < G'$ where $D = S^J$. Therefore, D/B and S/D are unramified in codimension one. Hence, by [10, Theorem 16.1] we have that $\operatorname{Ker}\theta \cong H^1(J, U_S)$ where $\theta: \mathscr{C}lD \to \mathscr{C}lS$ is the induced homomorphism. However, $\operatorname{Ker}\theta \cong \mathscr{C}lD$ since the map $\theta: \mathscr{C}lD \to \mathscr{C}lS$ factors $\theta = \beta\alpha$ where $\alpha: \mathscr{C}lD \to \mathscr{C}lA$ and $\beta: \mathscr{C}lA \to \mathscr{C}lS$. Of course $\beta = 0$ since A is factorial. Finally, from [13, Lemma 2.3] and the fact that $\mathscr{C}\ell D \cong H^1(J, U_S)$, we conclude that $\mathscr{C}\ell D = 0$ and D is factorial. Our result [13, Theorem 2.1] now completes our argument.

While the requirement of uniform ramification in codimension one together with elimination of inseparability in the extension of fraction fields would appear to improve the behavior of local factorial domains (i.e., they would seem to be less pathological than the examples in [5, 11, 12, 24]) it still remains to gain more control on the splitting of primes in codimension one. In Theorem 4.1 this was accomplished by assuming that the ramified primes in codimension one were normal. However, it would be more satisfying to work under a less restrictive hypothesis.

Note added in proof. The author has recently discovered that the main thrust of Theorem 2.4 appears in S. Abhyankar's article [25, Proposition 1].

References

1. S. Abhyankar, Ramification theoretic methods in algebraic geometry, *in* "Annals of Mathematics Studies," Vol. 43, Princeton Univ. Press, Princeton, NJ, 1959.
2. M. Auslander, On the purity of the branch locus, *Amer. J. Math.* **84** (1962), 116–125.
3. M. Auslander and D. Buchsbaum, On ramification theory in Noetherian rings, *Amer. J. Math.* **81** (1959), 749–765.
4. M. Auslander and O. Goldman, The Brauer group of a commutative ring, *Trans. Amer. Math. Soc.* **97** (1960), 367–409.
5. M.-J. Bertin, Anneaux d'invariants d'anneaux de polynômes en caractéristique *p*, *C. R. Acad. Sci. Paris* **264** (1967), 653–656.
6. N. Bourbaki, "Algèbre Commutative," Herman, Paris, 1965.
7. V. I. Danilov, Rings with a discrete group of divisor classes, *Math. USSR-Sb.* **12** (1970), 368–386 (*Mat. Sb. (N.S.)* **83** (1970), 372–389).
8. F. DeMeyer and E. Ingraham, Separable algebras over commutative rings, *in* "Lecture Notes in Math.," Vol. 181, Springer, Berlin/Heidelberg/New York, 1971.
9. E. G. Evans and P. Griffith, Syzygies, *in* "London Math. Soc. Lecture Note Series," Vol. 106, Cambridge Univ. Press, Cambridge, 1985.
10. R. Fossum, The divisor class group of a Krull domain, *in* "Ergebnisse der Mathematik und ihrer Grenzgebiete," Vol. 74, Springer, Berlin/Heidelberg/New York, 1973.
11. R. Fossum and P. Griffith, Complete local factorial rings which are not Cohen–Macaulay in characteristic *p*, *Ann. Sci. Ecole Norm Sup. (4)* **8** (1974), 189–200.
12. E. Freitag and R. Kiehl, Algebraische Eigenschaften der localen Ringe in den Spitzen de Hilbertschen Modulgruppen, *Invent. Math.* **24** (1974), 121–148.
13. P. Griffith, Normal extensions of regular local rings, *J. Algebra* **106** (1987), 465–475.
14. P. Griffith, Pathology in divisor class groups of hypersurfaces, *J. Pure Appl. Algebra* **41** (1986), 37–49.
15. A. Grothendieck, Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux, *in* "Séminaire de Géometrie Algébrique (SGA), 1962, IHES fasc. I and II, 1963."

16. A. GROTHENDIECK, Local cohomology, *in* "Lecture Notes in Math.," Vol. 41, Springer, Berlin/Heidelberg/New York, 1967.
17. R. HARTSHORNE AND A. OGUS, On the factoriality of local rings of small embedding codimension, *Comm. Algebra* **5**, No. 1 (1974), 415–437.
18. J. HERZOG AND E. KUNZ, Der kanonische Modul eines Cohen–Macaulay–Rings, *in* "Lecture Notes in Math.," Vol. 238, Springer, Berlin/Heidelberg/New York, 1971.
19. H. MATSUMARA, "Commutative Algebra," Benjamin, New York, 1970.
20. M. NAGATA, Local rings, *in* "Interscience Tracts in Pure and Applied Mathematics," Vol. 13, Wiley, New York, 1962.
21. P. ROBERTS, Abelian extensions of regular local rings, *Proc. Amer. Math. Soc.* **78** (1980), 307–310.
22. P. SAMUEL, On unique factorization domains, *Illinois J. Math.* **5** (1961), 1–17.
23. J.-P. SERRE, Algèbre locale—Multiplicities, *in* "Lecture Notes in Math.," Vol. 11, Springer, Berlin/Heidelberg/New York, 1965.
24. D. WESTON, Generating absolute prime ideals, submitted for publication.
25. S. ABHYANKAR, Tame coverings and fundamental groups of algebraic varieties, *Amer. J. Math.* **82** (1960), 179–189.